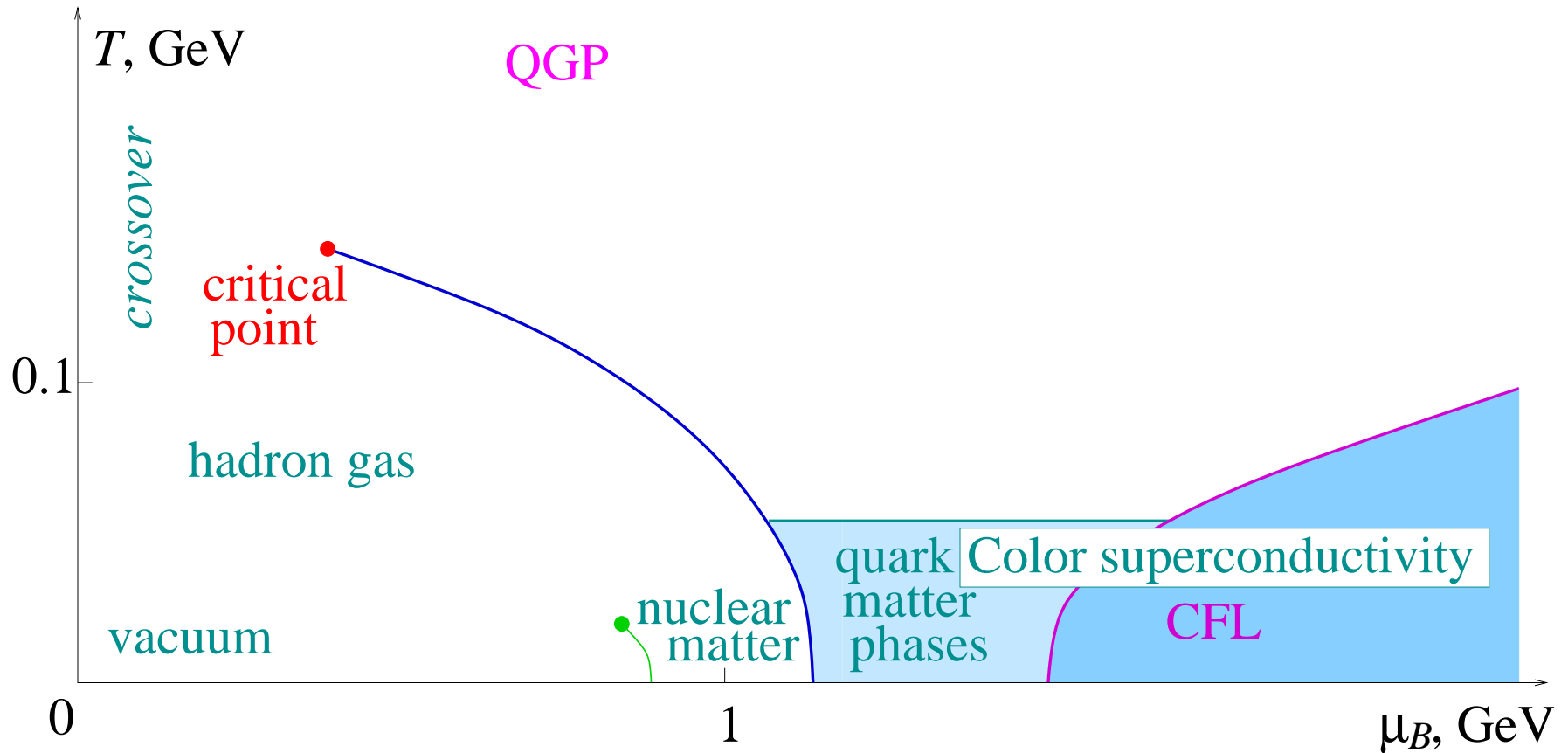


# Universality, non-Gaussian fluctuations and the search for the QCD critical point

M. Stephanov

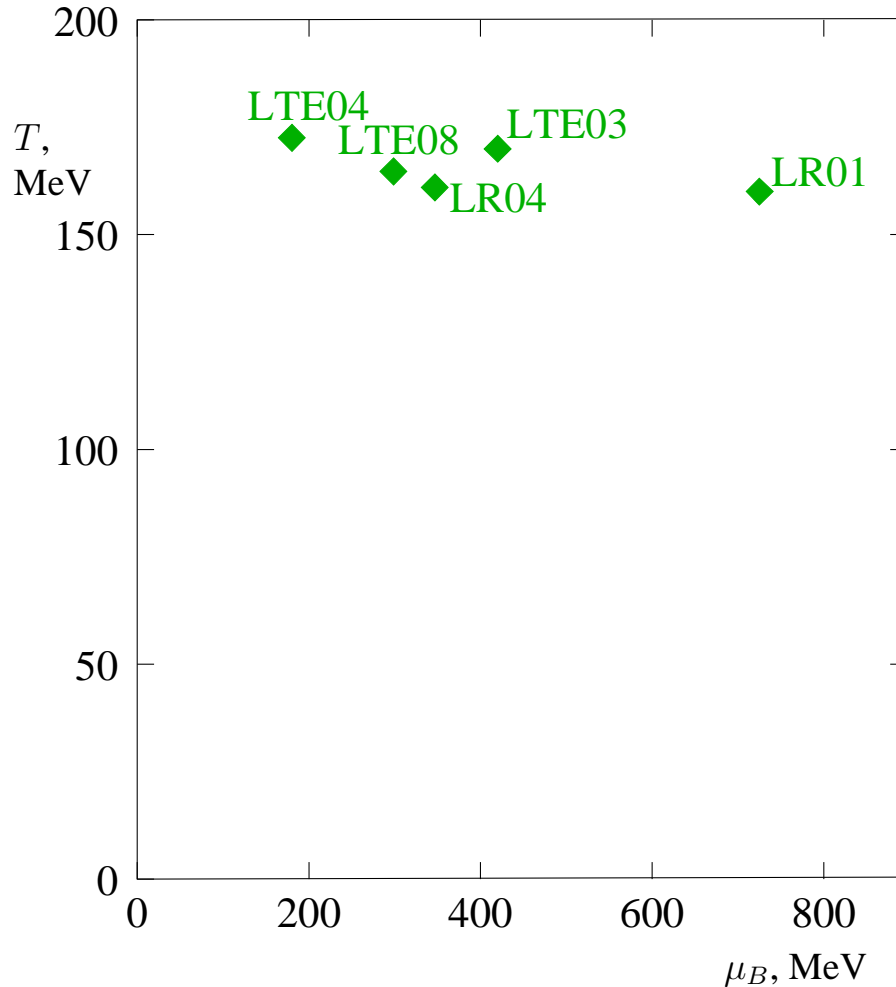
*U. of Illinois at Chicago*

# QCD phase diagram (contemporary view)



● Models (and lattice) suggest crossover turns into 1st order at some  $\mu_B$ .

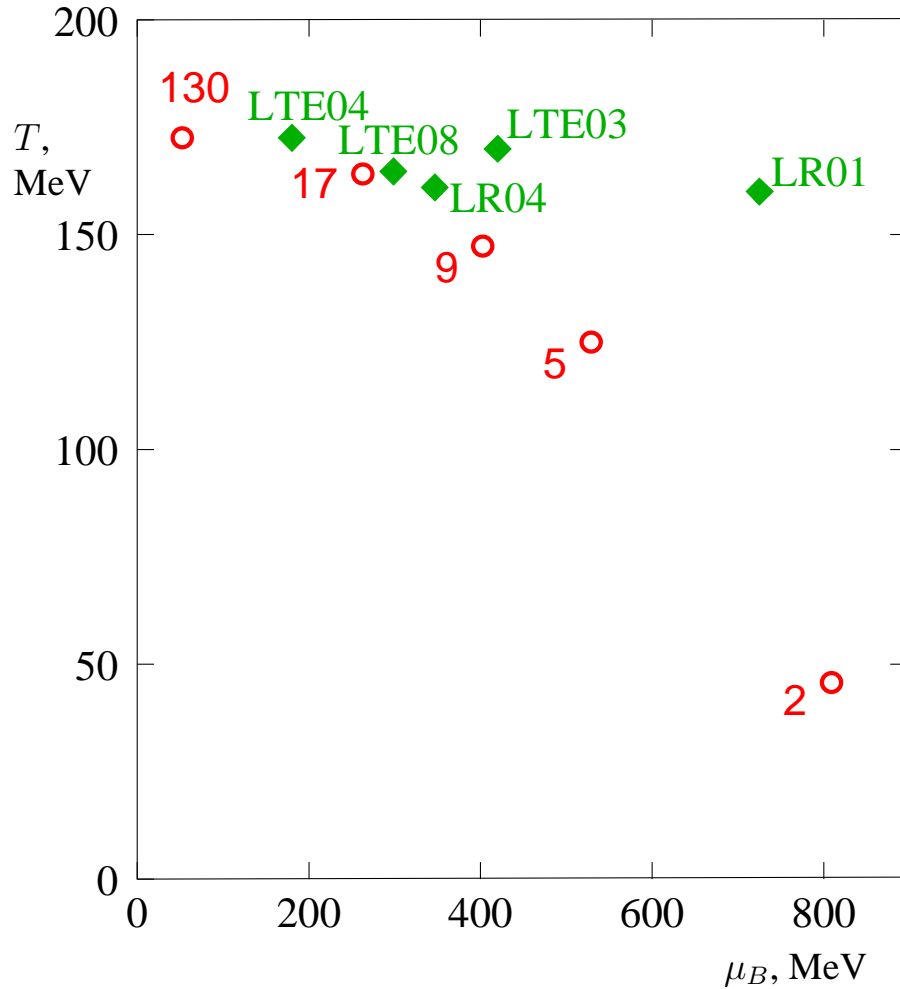
# Location of the critical point vs freeze-out



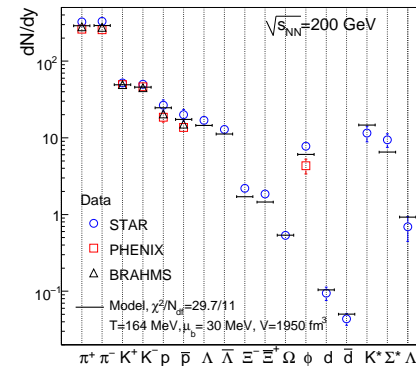
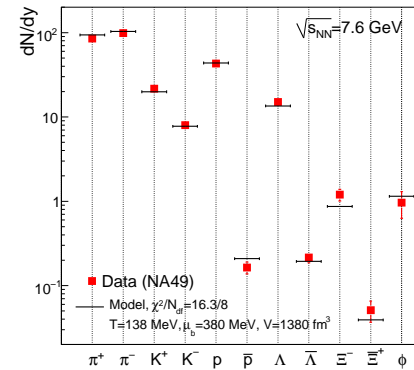
Estimates from lattice MC:

- Systematic errors are not shown.
  - So far lattice results *disfavor*  $\mu_B < 200$  MeV.
  - de Forcrand-Philipsen: maybe  $\mu_B > 500$  MeV?
  - Strong lat. spacing dependence:
    - continuum limit is still far?
    - role of anomaly and “rooting”?
- Wilson fermions might help.

# Location of the critical point vs freeze-out

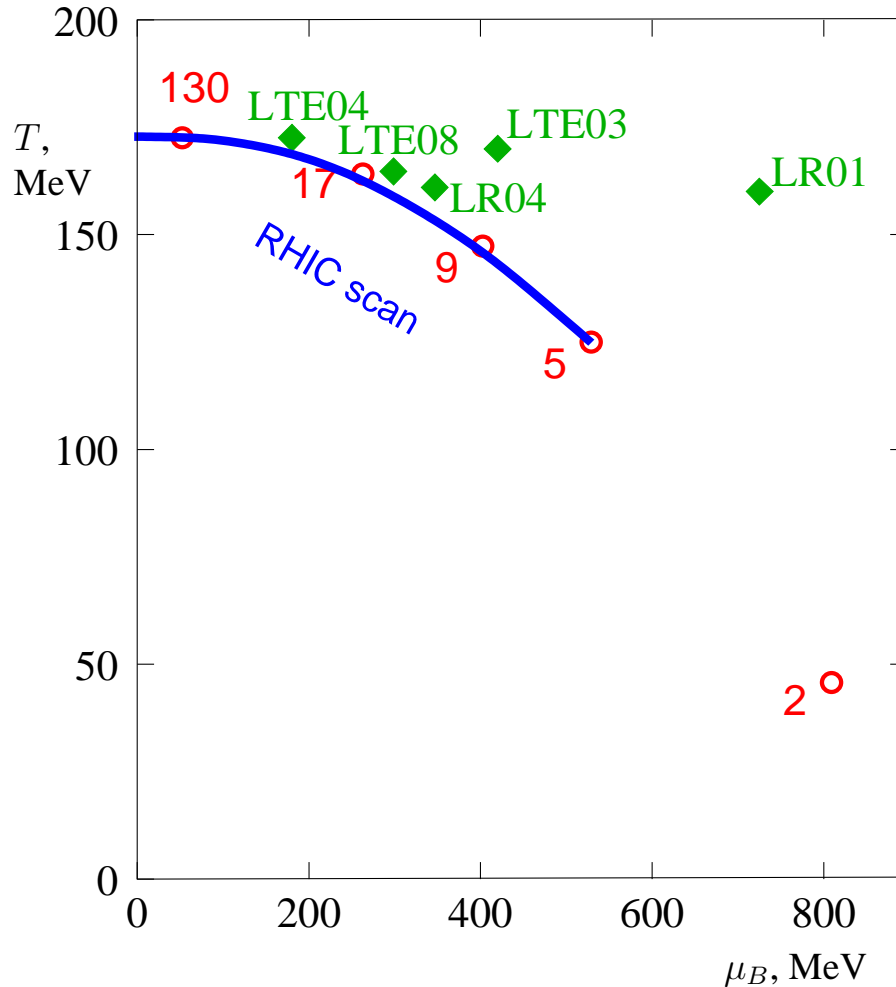


● Final state is thermal



Thermal model 2008  
(Andronic-PBM-Stachel)

# Location of the critical point vs freeze-out



To discover the critical point and establish its location, one needs:

● Energy-scan experiments:

● RHIC,

● NA61(SHINE) @ SPS,

● CBM @ FAIR/GSI,

● NICA @ JINR

● Improve lattice predictions  
– understand systematic errors

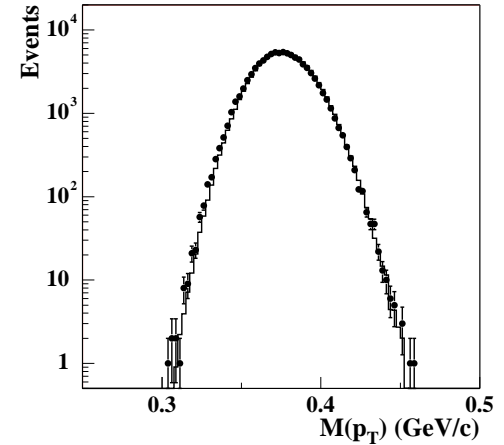
● Understand critical phenomena in the dynamical environment of a h.i.c. (understand background)  
– develop optimal signatures

# Talk summary

- Experiments measure for each event: multiplicities  $N_\pi, N_p, \dots$ , momenta  $p$ , etc. These quantities fluctuate event-by-event.

- Typical measure is stdev, e.g.,  $\langle(\delta N)^2\rangle$ .

- What is the magnitude of these fluctuations near the c.p.? (Rajagopal, Shuryak, M.S.)



- Universality* tells how it grows at the critical point:  $\langle(\delta N)^2\rangle \sim \xi^2$ . Correlation length is a universal measure of the “distance” from the c.p. It diverges as  $\xi \sim (\Delta\mu \text{ or } \Delta T)^{-2/5}$  as the c.p. is approached.
  - Magnitude of  $\xi$  is limited  $< \mathcal{O}(3 \text{ fm})$  (Berdnikov, Rajagopal).
- 

- “Shape” of the fluctuations can be measured: non-Gaussian moments. As  $\xi \rightarrow \infty$  fluctuations become less Gaussian ( $1/N$  effect).
- Higher cumulants show even stronger dependence on  $\xi$  (PRL 102:032301,2009):

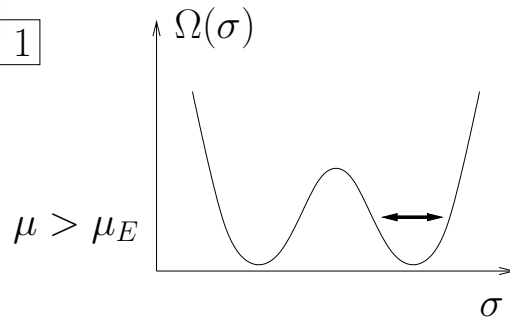
$$\langle(\delta N)^3\rangle \sim \xi^{4.5}, \quad \langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2 \sim \xi^7$$

which makes them more sensitive signatures of the critical point.

# Critical mode and equilibrium fluctuations



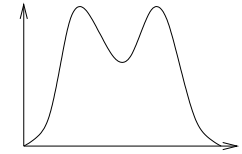
1



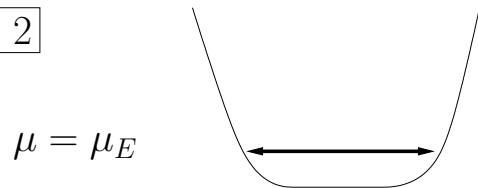
$$\bar{\psi}\psi - \langle \bar{\psi}\psi \rangle \equiv \sigma$$

$$\langle \sigma^2 \rangle \sim (\Omega'')^{-1}$$

think  $e^{-\Omega}$ :

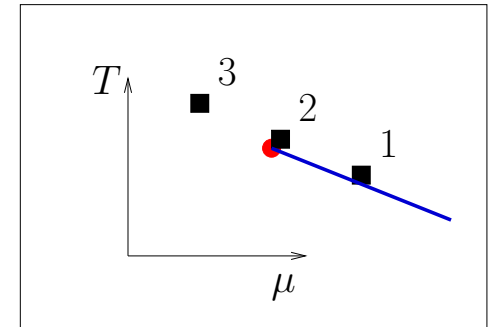


2

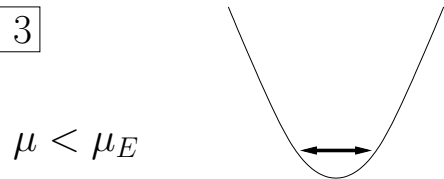


$$(\Omega'')^{-1} \rightarrow \infty$$

large **equilibrium** fluctuations



3



Magnitude of fluctuation and correlation length:

$$\langle \sigma(\mathbf{x})\sigma(\mathbf{0}) \rangle \sim \begin{cases} e^{-|\mathbf{x}|/\xi} & \text{for } |\mathbf{x}| \gg \xi \\ 1/|\mathbf{x}| & \text{for } |\mathbf{x}| \ll \xi \end{cases}$$

$$\langle \sigma_0^2 \rangle = \int d^3x \langle \sigma(\mathbf{x})\sigma(\mathbf{0}) \rangle \sim \xi^2$$

critical singularity is a *collective* phenomenon

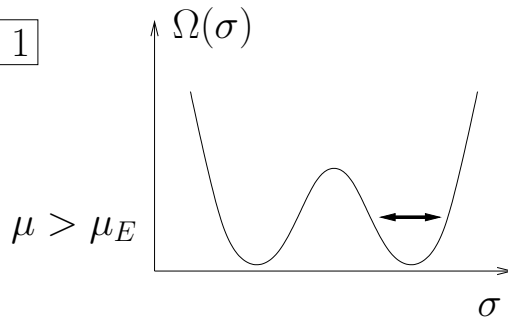


$\sigma$  or  $n_B$  or  $T^{00}$ ? Because they mix, only *one* linear combination is critical.

# Critical mode and equilibrium fluctuations



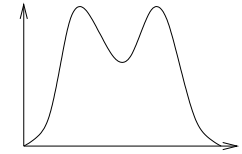
1



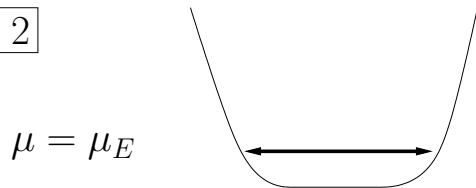
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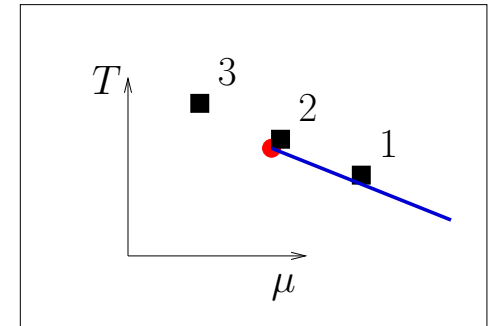


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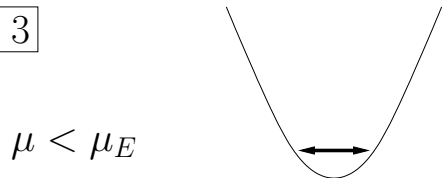


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Magnitude of fluctuation and correlation length:

$$\langle \sigma(\mathbf{x}) \sigma(\mathbf{0}) \rangle \sim \begin{cases} e^{-|\mathbf{x}|/\xi} & \text{for } |\mathbf{x}| \gg \xi \\ 1/|\mathbf{x}|^{1+\eta} & \text{for } |\mathbf{x}| \ll \xi \end{cases}$$

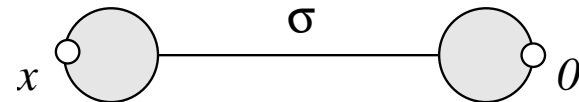
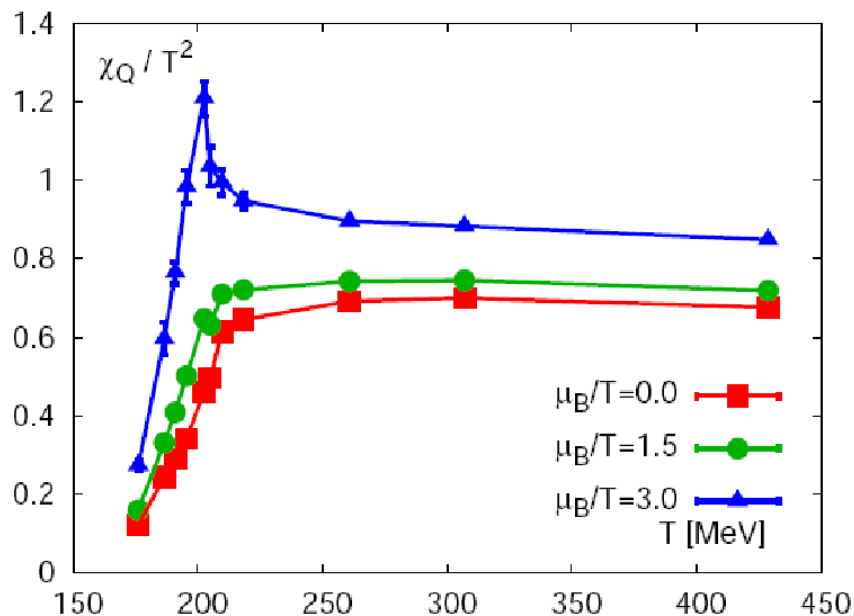
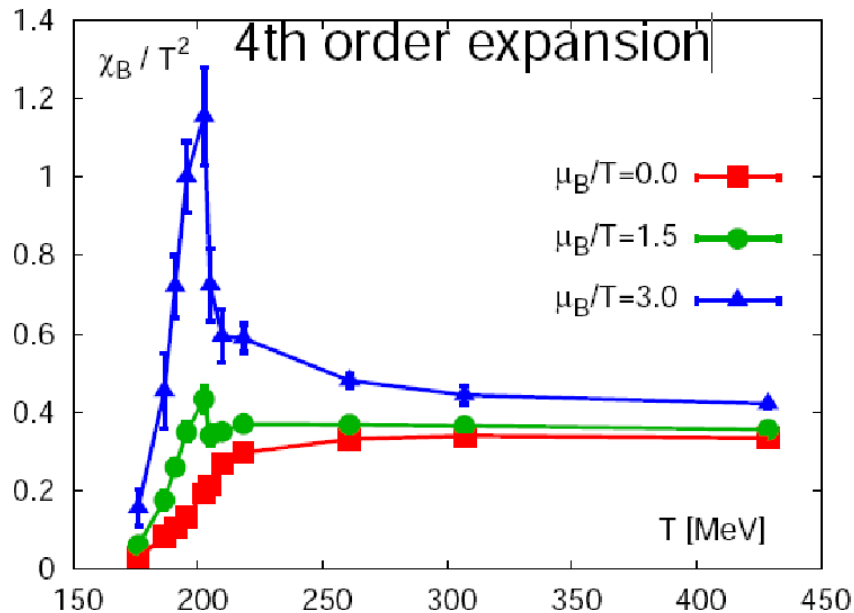
$$\langle \sigma_0^2 \rangle = \int d^3 \mathbf{x} \langle \sigma(\mathbf{x}) \sigma(\mathbf{0}) \rangle \sim \xi^{2-\eta}$$

critical singularity is a *collective* phenomenon



$\sigma$  or  $n_B$  or  $T^{00}$ ? Because they mix, only *one* linear combination is critical.

# Fluctuations on the lattice



Baryon charge density - isoscalar.

Electric charge  $Q = I_3 + B/2$ .

No peak in isospin (nonsinglet) susceptibility.

# Relation between $\sigma$ fluctuations and observables

Consider example: fluctuations of multiplicity of pions (or protons).

- Free gas:  $n_p^0$  – fluctuating occupation number of momentum mode  $p$ .  
Ensemble (event) average  $\langle n_p^0 \rangle = f_p$  and

$$n_p^0 = f_p + \delta n_p^0; \quad \langle \delta n_p^0 \delta n_k^0 \rangle = f'_p \delta_{pk}; \quad f_p = (e^{\omega_p/T} \mp 1)^{-1}; \quad f'_p \equiv f_p(1 \pm f_p).$$

- Couple these particles to  $\sigma$  field:  $G\sigma\pi\pi$  (or  $g\sigma\bar{N}N$ ).  
Think of  $m^2 \equiv m_0^2 + 2G\sigma$  as “fluctuating mass”. Then

$$\delta n_p = \delta n_p^0 + \frac{\partial f_p}{\partial m^2} 2G\sigma = \delta n_p^0 + \frac{f'_p}{\omega_p} \frac{G}{T} \sigma$$

- Using  $\langle \delta n_p^0 \sigma \rangle = 0$  and  $\langle \sigma^2 \rangle = (T/V)\xi^2$ .

$$\langle \delta n_p \delta n_k \rangle = f'_p \delta_{pk} + \frac{1}{VT} \frac{f'_p}{\omega_p} \frac{f'_k}{\omega_k} G^2 \xi^2.$$

More formal derivation: PRD65:096008,2002

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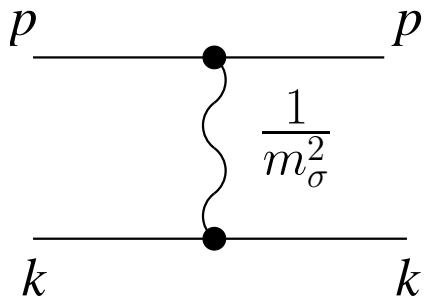
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More formal derivation: PRD65:096008,2002

# 4-point function

- The 2-particle correlator measures 4-point function at  $q = 0$  (for  $p \neq k$ ). Singularity appears at  $q = 0$  due to vanishing  $\sigma$  screening mass  $m_\sigma \rightarrow 0$ . (i.e.,  $\xi = 1/m_\sigma \rightarrow \infty$ ).



$$\langle \delta n_p \delta n_k \rangle_\sigma = \frac{1}{T} \frac{f_p(1+f_p)}{\omega_p} \frac{f_k(1+f_k)}{\omega_k} \frac{G^2}{m_\sigma^2}.$$

Check:  $\langle \delta n_p \delta n_k \rangle = \langle n_p n_k \rangle - \langle n_p \rangle \langle n_k \rangle > 0$  — as in attraction. Attraction lowers the energy of a pair (making it more likely) by  $\langle H_{\text{interaction}} \rangle \sim$  forward scattering amplitude.

- Consider baryon number susceptibility, which should diverge:  $\chi_B \sim \xi^{2-\eta}$

$$\chi_B \sim \langle \delta B \delta B \rangle_\sigma = \langle (\delta N_p - \delta N_{\bar{p}} + \delta N_n - \delta N_{\bar{n}})^2 \rangle_\sigma = \langle \delta N_p \delta N_p \rangle_\sigma + \dots$$

Each term on r.h.s. is  $\sim \frac{1}{m_\sigma^2}, \quad \Rightarrow \quad \langle \delta B \delta B \rangle \sim 1/m_\sigma^2 = \xi^2.$

- ● It is enough to measure protons  $\langle \delta N_p \delta N_p \rangle$  (Hatta, MS, PRL91:102003,2003)

# Limitations on $\xi$ in heavy-ion collisions

How big can  $\xi$  grow?

Limited by:

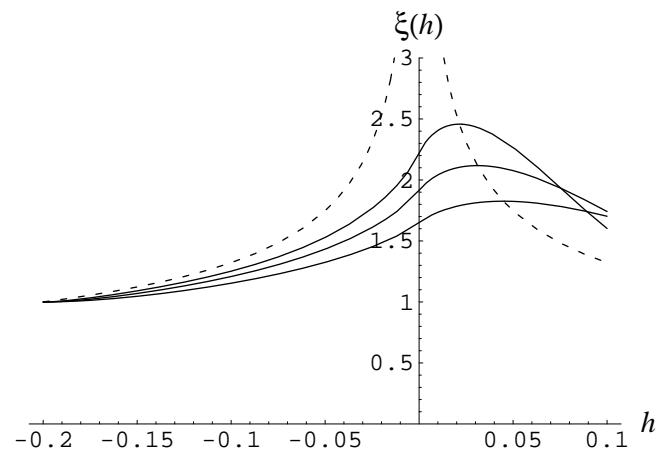
- Proximity of the critical point
- Finite size of the system  $\xi < 6$  fm.
- Finite *time*:  $\tau \sim 10$  fm.

Critical slowing down:  $\tau_{\text{equilibration}} \sim \xi^z$ .  
 $z > 1$  – dynamical critical exponent.

$$\xi_{\text{max}} \sim \tau^{1/z} \sim (2 - 3)\text{fm}$$

Dynamic universality class of liquid-gas phase transition, i.e.,  $z \approx 3$ :

- Critical mode – diffusive:  $\omega \sim iDq^2$ ,
- $D = \frac{\lambda_B}{\chi_B} \rightarrow 0$  at c.p.  $2 + 1 = 3$ .  
 (Son, MS, PRD70:056001,2004)



(Berdnikov, Rajagopal,  
PRD61:105017,2000)

# Higher moments (cumulants) of fluctuations

- Consider probability distribution for the order-parameter field:

$$P[\sigma] \sim \exp \{ -\Omega[\sigma]/T \} ,$$

$\Omega$  – effective potential:

$$\Omega = \int d^3x \left[ \frac{1}{2} (\nabla \sigma)^2 + \frac{m_\sigma^2}{2} \sigma^2 + \frac{\lambda_3}{3} \sigma^3 + \frac{\lambda_4}{4} \sigma^4 + \dots \right] . \quad \Rightarrow \quad \xi = m_\sigma^{-1}$$

- Moments of zero-momentum mode  $\sigma_0 \equiv \int d^3x \sigma(x)/V$ .

$$\kappa_2 = \langle \sigma_0^2 \rangle = \frac{T}{V} \xi^2 ; \quad \kappa_3 = \langle \sigma_0^3 \rangle = \frac{2\lambda_3 T^2}{V^2} \xi^6 ;$$

$$\kappa_4 = \langle \sigma_0^4 \rangle_c \equiv \langle \sigma_0^4 \rangle - \langle \sigma_0^2 \rangle^2 = \frac{6T^3}{V^3} [2(\lambda_3 \xi)^2 - \lambda_4] \xi^8 .$$

- Tree graphs. Each zero-momentum propagator gives  $m_\sigma^{-2}$ , i.e.,  $\xi^2$ .



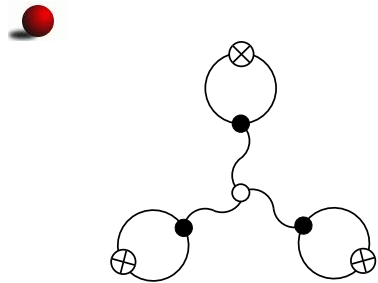
# Moments of *observables*

- Use multiplicity for an example. Since multiplicity is just the sum of all occupation numbers, and thus

$$\delta N = \sum_{\mathbf{p}} \delta n_{\mathbf{p}},$$

the cubic moment (skewness) of the pion multiplicity distribution is given by

$$\langle (\delta N)^3 \rangle = \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{\mathbf{p}_3} \langle \delta n_{\mathbf{p}_1} \delta n_{\mathbf{p}_2} \delta n_{\mathbf{p}_3} \rangle, \quad \text{where } \sum_{\mathbf{p}} = V \int d^3 \mathbf{p} / (2\pi)^3.$$



$$\langle \delta n_{\mathbf{p}_1} \delta n_{\mathbf{p}_2} \delta n_{\mathbf{p}_3} \rangle_{\sigma} = \frac{2\lambda_3}{V^2 T} \left( \frac{G}{m_{\sigma}^2} \right)^3 \frac{v_{\mathbf{p}_1}^2}{\omega_{\mathbf{p}_1}} \frac{v_{\mathbf{p}_2}^2}{\omega_{\mathbf{p}_2}} \frac{v_{\mathbf{p}_3}^2}{\omega_{\mathbf{p}_3}}$$

$$v_{\mathbf{p}}^2 = \bar{n}_{\mathbf{p}} (1 \pm \bar{n}_{\mathbf{p}})$$

Similarly for  $\langle (\delta N)^4 \rangle_c$ .

- Since  $\langle (\delta N)^3 \rangle$  scales as  $V^1$  it is convenient to normalize it by the mean total multiplicity  $\bar{N}$  which scales similarly. Thus we define

$$\omega_3(N) \equiv \frac{\langle (\delta N)^3 \rangle}{\bar{N}}$$

# Moments of observables contd.

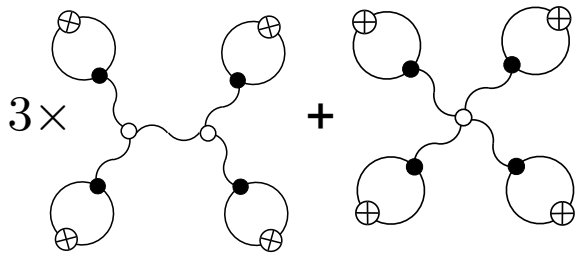
... and find

$$\omega_3(N)_\sigma = \frac{2\lambda_3}{T} \frac{G^3}{m_\sigma^6} \left( \int_p \frac{v_p^2}{\omega_p} \right)^3 \left( \int_p \bar{n}_p \right)^{-1}.$$

🟢 Similarly, for

$$\omega_4(N) \equiv \frac{\langle (\delta N)^4 \rangle_c}{\bar{N}}$$

from



we find

$$\omega_4(N)_\sigma = \frac{6}{T} \left[ 2 \frac{\lambda_3^2}{m_\sigma^2} - \lambda_4 \right] \frac{G^4}{m_\sigma^8} \left( \int_p \frac{v_p^2}{\omega_p} \right)^4 \left( \int_p \bar{n}_p \right)^{-1}.$$

# Moments of observables contd.

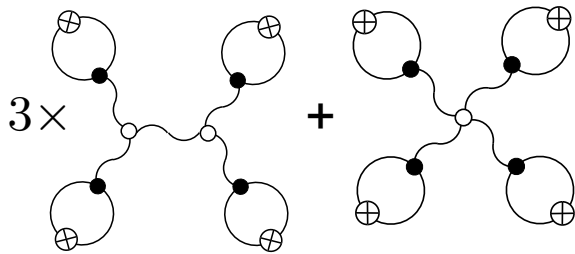
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# Scaling, $\lambda_n$

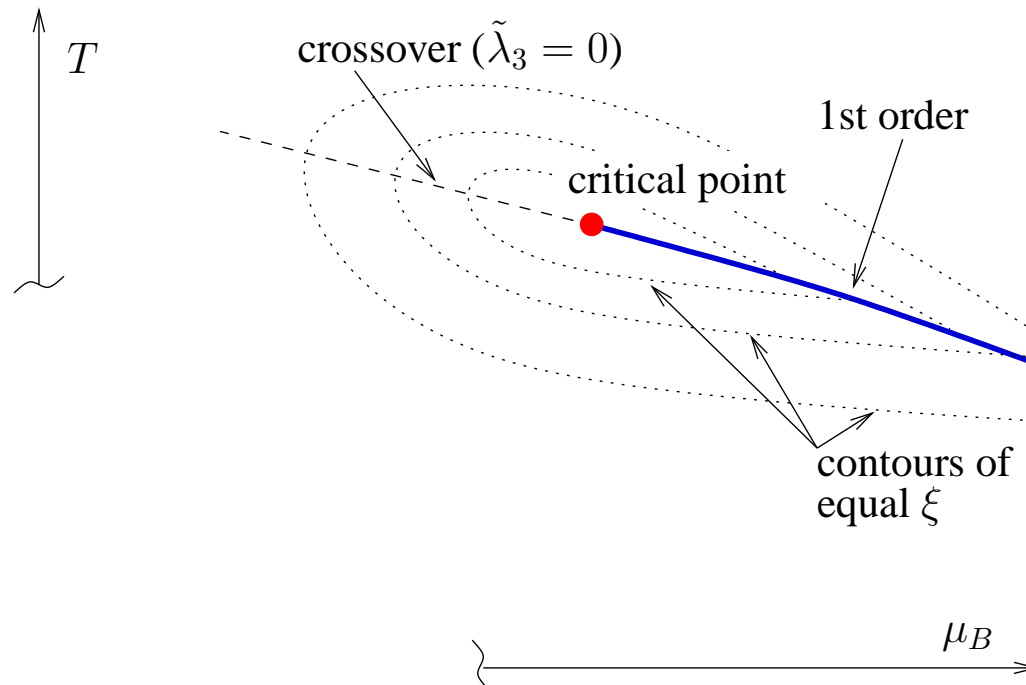
- Scaling requires that both  $\lambda_3$  and  $\lambda_4$  vanish with a power of  $\xi$  given by:

$$\lambda_3 = \tilde{\lambda}_3 T \cdot (T\xi)^{-3/2}, \quad \text{and} \quad \lambda_4 = \tilde{\lambda}_4 \cdot (T\xi)^{-1}, \quad (\eta \ll 1)$$

(because  $[(\nabla\sigma)^2] = 3 \Rightarrow [\sigma] = 1/2$  and  $\Rightarrow [\lambda_n] = 3 - n/2$ )

Dimensionless couplings  $\tilde{\lambda}_3$  and  $\tilde{\lambda}_4$  are universal, and for the Ising universality class they have been measured on the lattice.

- $\lambda_3$  is nonzero:



# Scaling, $\lambda_n$

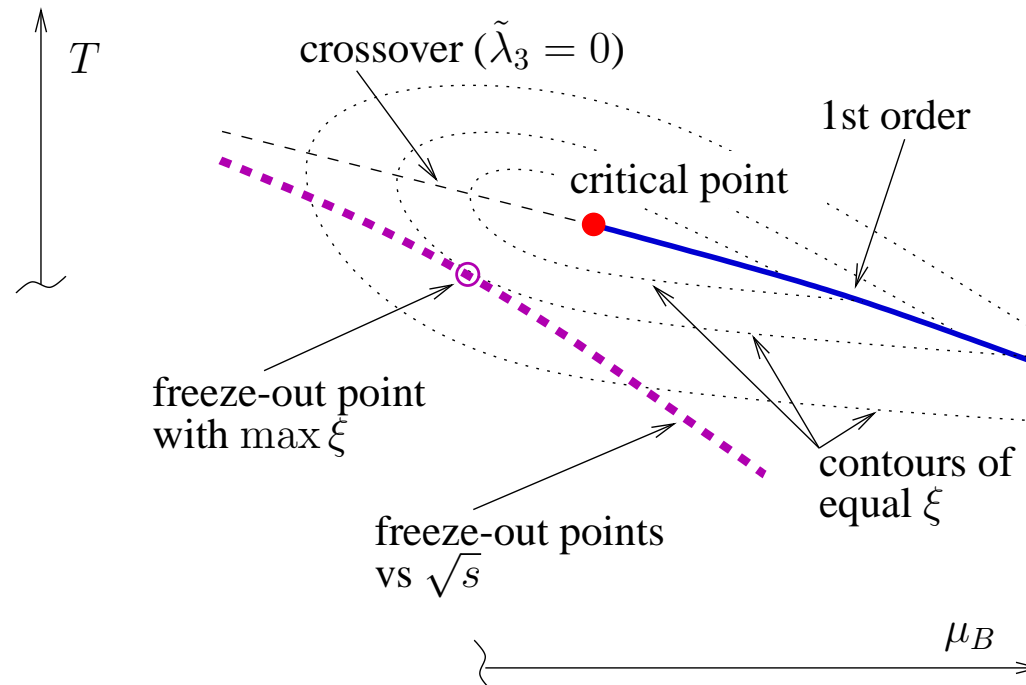
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# Estimates

Pions (top SPS):

$$\omega_3(N_\pi)_\sigma \equiv \frac{\langle (\delta N_\pi)^3 \rangle}{\bar{N}_\pi} \approx 1. \left( \frac{\tilde{\lambda}_3}{4.} \right) \left( \frac{G}{300 \text{ MeV}} \right)^3 \left( \frac{\xi}{3 \text{ fm}} \right)^{9/2}$$

$$\omega_4(N_\pi)_\sigma \equiv \frac{\langle (\delta N_\pi)^4 \rangle_c}{\bar{N}_\pi} \approx 12. \left( \frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.} \right) \left( \frac{G}{300 \text{ MeV}} \right)^4 \left( \frac{\xi}{3 \text{ fm}} \right)^7$$

Protons (top SPS):

$$\omega_3(N_p)_\sigma \equiv \frac{\langle (\delta N_p)^3 \rangle}{\bar{N}_p} \approx 3. \left( \frac{\tilde{\lambda}_3}{4.} \right) \left( \frac{g}{10.} \right)^3 \left( \frac{\xi}{1 \text{ fm}} \right)^{9/2}$$

$$\omega_4(N_p)_\sigma \equiv \frac{\langle (\delta N_p)^4 \rangle_c}{\bar{N}_p} \approx 23. \left( \frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.} \right) \left( \frac{g}{10.} \right)^4 \left( \frac{\xi}{1 \text{ fm}} \right)^7$$

## Notes:

- Strong dependence on  $\xi$ , compared to  $\omega_2 \sim \xi^2$ .
- Significant uncertainty due to  $G$ ,  $g$ .
- Crosscheck: same exponents as baryon number cumulants from scaling/universality:

$$\langle (\delta N_B)^k \rangle_c = VT^{k-1} \frac{\partial^k P(T, \mu_B)}{\partial \mu_B^k} \sim \xi^{k(5-\eta)/2-3}. \quad (\eta \ll 1)$$

# Concluding remarks I

- Sign of  $\omega_3$ ? *Positive* for  $N_\pi$  and  $N_p$ .

Crude argument:

(a)  $N_\pi$  and  $N_p$  are proxies for  $s$  and  $n_B$ , and

(b) e.g.,  $\langle (\delta S)^3 \rangle = T^2 \frac{d^2 S}{dT^2} > 0$  below C.P. because  $\frac{dS}{dT}$  peaks (Asakawa *et al*).

- Trivial background estimate:  $\omega_3(N)_{\text{BE}} = \overline{(1 + n_p)(1 + 2n_p)}$ .

This is about 1.3 ( $\approx 1 + 3 \overline{n_p}$ ) for pions at  $T = 120$  MeV

$$\omega_4(N)_{\text{BE/FD}} = \overline{(1 \pm n_p)(1 \pm 6n_p(1 \pm n_p))}$$

- Note:  $\omega_k(N_{\pi+} + N_{\pi-})_{\text{BE}} = \omega_k(N_{\pi+})_{\text{BE}} - \text{i.e., no cross-correlation.}$

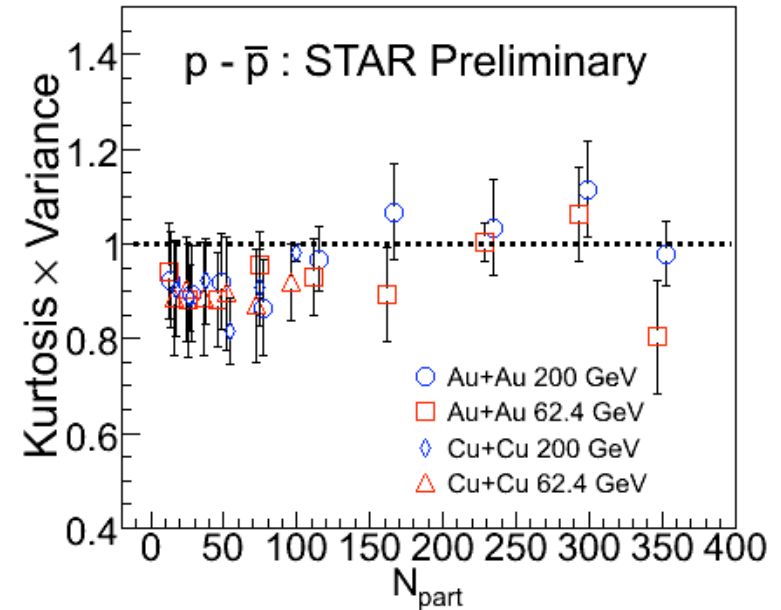
In contrast, for the critical point contribution:

$$\omega_k(N_{\pi+} + N_{\pi-})_\sigma = 2^{k-1} \omega_k(N_{\pi+})_\sigma.$$

# Concluding remarks II

- Measuring  $\omega_3$  maybe be harder (more statistics needed?) than  $\omega_2$ .  
More so for  $\omega_4 \sim \langle (\delta N)^4 \rangle - 3\langle (\delta N)^2 \rangle^2 = \mathcal{O}(N^2) - \mathcal{O}(N^2) \sim \mathcal{O}(N^1)$ .
- Other, non-critical, sources contribute: remnants of initial fluctuations, flow, jets – to name just a few.

- Calculate and subtract background.
- Apply kinematic cuts. (Low pt.)
- Measure bkgnd during the energy scan.  
STAR: background is small  $\omega_4 \approx 1$  (Poisson).



- Non-Gaussian moments have stronger dependence on  $\xi$ , and thus are more sensitive signatures of the critical point.